Solutions for Midterm Exam

MAS501 Analysis for Engineers, Spring 2011

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1. (a) True.
   (b) True.
   (c) True.
   (d) False.
   (e) True.
   (f) True.
   (g) False.
   (h) False.
   (i) True.
   (j) True.

2. (a) See Problem 1.3.7 in the textbook.
   (b) Done in the class; see Problem 2.4.2 in the textbook.

3. Because the domain Ω is compact, f is uniformly continuous on Ω by theorem 4.2.4. Hence the problem is reduced to the second problem in Homework 5.

4. (a) The Heine-Borel theorem does not hold for metric spaces in general.
   (b) Give the discrete metric to a set of natural numbers $\mathbf{N}$. (For the definition of discrete metric, refer to Homework 2.) Consider a sequence $\{x_n = n\}$, then $d(1, x_n) \leq 1$ by definition of discrete metric, hence the sequence $\{x_n\}$ is bounded. Now assume there is a subsequence $\{x_{n_i}\}$ converging to a point $x \in \mathbf{N}$. Then we have $d(x, x_{n_i}) \to 0$ as $i \to \infty$. Hence

   $$d(x, x_{n_i}) < \frac{1}{2} \quad \text{eventually.}$$

   By definition of discrete metric, this means that $x = x_{n_i} = n_i$ eventually; this is obviously impossible. Hence the *generalized* Bolzano-Weierstrass theorem does not hold.

5. Assume the contrary:

   $$\limsup_{n\to\infty} \left(\frac{a_{n+1}+1}{a_n}\right)^n < 1.$$

   Then we have

   $$\left(\frac{a_{n+1}+1}{a_n}\right)^n < 1 \quad \text{eventually.}$$

   Hence it holds that

   $$a_{n+1} < a_n - 1 \quad \text{eventually,}$$

   i.e., there is a $N > 0$ such that for every $n \geq N$, we have

   $$a_{n+1} < a_n - 1.$$

Now we can easily verify that for any $n \geq 0$,

$$a_{N+n} < a_N - n.$$

But this means that for sufficiently large n, we have $a_{N+n} < 0$; which contradicts to the assumption that every a_n is positive.

Note: In fact, it holds that

$$\limsup_{n\to\infty} \left(\frac{a_{n+1}+1}{a_n}\right)^n \geq e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n > 2.718$$

and the inequality need not hold with e replaced by a larger number. This result is attributed to a Hungarian mathematician George Pólya (1887–1985).

6. It suffices to show that for every real numbers a, b such that $0 < a < b < \infty$, it holds that

$$\frac{f(b)}{b} - \frac{f(a)}{a} \geq 0.$$

And this inequality is equivalent to $af(b) - bf(a) \geq 0$. Let $0 < a < b < \infty$. Then by the mean value theorem, there is a $x \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(x).$$

Because $f'(x)$ is increasing on $(0, \infty)$, we have

$$\frac{f(b)-f(a)}{b-a} \geq f'(a).$$

On the other hand, also by the mean value theorem, there is a $y \in (0, a)$ such that

$$\frac{f(a)}{a} = \frac{f(a)-f(0)}{a-0} = f'(y).$$

By monotonicity of f' again, we have

$$\frac{f(a)}{a} \leq f'(a).$$

Therefore it holds that

$$\frac{f(b)-f(a)}{b-a} \geq \frac{f(a)}{a}.$$

But this implies that

$$\frac{f(b)-f(a)}{b-a} - \frac{f(a)}{a} = \frac{af(b)-bf(a)}{a(b-a)} \geq 0.$$

Hence $af(b) - bf(a) \geq 0$ and the proof is complete.

Another proof: It suffices to show that

$$\frac{d}{dx}\frac{f(x)}{x} \geq 0 \quad \text{for every } x > 0,$$

which is equivalent to show that

$$f'(x) \geq \frac{f(x)}{x} \quad \text{for every } x > 0.$$

And we can verify that this is true by the mean value theorem and monotonicity of f' (as in the previous proof).

Bonus! Because the set of rational numbers is countable, we can list them as a sequence $\{r_n\}$. Define a function $g_n : \mathbf{R} \to \mathbf{R}$ as

$$g_n(x) := \begin{cases} 0 & \text{if } x < r_n, \\ 1 & \text{if } x \geq r_n. \end{cases}$$

Then

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{2^n}\, g_n(x)$$

is an increasing function whose set of discontinuities is precisely $\mathbf{Q}$.